

# Partially coherent matter wave and its evolution

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## Abstract

The evolution and propagation of a partially coherent matter wave (PCMW) is investigated theoretically by the correlation function method. The ABCD matrix formalism previously used for a fully coherent matter wave is extended to the PCMW domain. A new ABCD law is derived, using a tensor method to describe the evolution of a PCMW. An analytical solution of the first-order correlation function is obtained that makes the propagation and evolution of a PCMW very simple and clear. As an example, the propagation of a PCMW in a gravitational field is calculated numerically.

## 1 Introduction

Matter-wave optics is a new and fascinating branch of research benefiting from the rapid progress of techniques for laser cooling and trapping of neutral atoms and molecules. The achievement of Bose-Einstein condensation (BEC) in dilute atomic and molecular gases enables the study of matter waves from incoherent to coherent domains [1]. In recent years, coherent matter waves have been studied extensively, such as the formation of vortices in BEC, linear and nonlinear propagation of atomic laser beams [4, 3, 2], and other phenomena. The matter wave has unique applications, as in atomic interferometry [5], atomic holography [6, 7], and so on.

According to existing theory, pure BEC exists only at zero temperature. However, zero temperature is not reachable experimentally due to the limits of laser cooling [8]. All the BEC and atomic lasers employed in experiments have distributions of momenta [2]. Therefore, matter waves, in the strict sense, are not completely coherent. All the cold atom gases produced by laser cooling at finite temperature are partially coherent matter waves (PCMW), whose properties during propagation and evolution are important issues and play a crucial role in the applications of matter waves. Nevertheless, they not been investigated sufficiently.

It is well known that a coherent matter wave can be described by a single wave function whose evolution is governed by the Gross Pitaevskii (GP) equation [2]. On the other hand, a PCMW can't be described by a single-particle wave function, but requires a much more complicated treatment. One method is the finite temperature BEC theory, which separates the matter wave field operator into two terms; one term describes the condensate part, and the other term deals with the noncondensate part. However, the equations based on this method are complicated and can't be solved analytically. Additional approximations are necessary to solve the equations. For example, in the Hartree-Fock-Bogliubov (HFB) approximation [9], the condensate wave function satisfies the GP equation, and the noncondensate operator is described by an equation that follows from subtracting the GP equation from the total field operator's Heisenberg equation. In the Hartree-Fock (HF) and Popov approximation [10], the condensate wave function is defined by the modified GP equation, whereby the "anomalous" density is neglected and the noncondensate part consists of thermally excited atoms as described in terms of a semiclassical phase space distribution function. In an extended HFB theory, the collisionless noncondensate dynamics is included within second-order perturbation schemes, and the expressions for damping rates and frequency shifts of low-energy modes are derived [11].

Another approach to describe a PCMW is the correlation function, which provides quantitative information about the coherence and the intensity of matter waves. The first-order correlation function characterizes local fluctuations of the phase of the complex matter wave field amplitude, and is related to the contrast achievable in an interference experiment. The second-order correlation function is related to fluctuations of the modulus of the complex matter wave field amplitude, and expresses the tendency of atoms either to cluster or to remain spatially separated [12]. Meystre et al. derived the Van-Citter-Zernike theorem of PCMW, which disregards the interactions among the particles. They analyzed the propagation of the matter wave from an incoherent matter wave source [13], and discussed the detection of the correlation function [14]. This method is analogous to that used in traditional optics [15, 16, 17]. In this paper, we focus on the propagation and evolution of PCMW by applying the correlation function method, and disregard the interactions among the particles, although it is not necessary. The ABCD matrix formalism [18, 19], which previously had only been applicable for the wave function of a single particle, is extended to the PCMW. A generalized ABCD law to describe the evolution of the PCMW is derived by a tensor method. An analytical solution of the first-order correlation function after evolution is obtained, which makes the evolution problem of the PCMW

very simple and clear. The results are useful in the analysis of the space-time coherence, spatial distribution, propagation and imaging properties of the PCMW.

This paper is organized as follows: In Sec.2, we introduce briefly the ABCD matrix formalism for a single-particle wavepacket. In Sec.3, the ABCD matrix formalism is extended by a tensor method to describe the PCMW. In Sec.4, the evolution properties of an ultracold atomic sample released in a gravitational field are calculated numerically and discussed. A summary is given in the Conclusion.

## 2 The ABCD matrix formalism

First, let us briefly describe the ABCD matrix formalism for a matter wave, introduced by Ch. J. Bordé based on the time-dependent Schrödinger equation [18], which can be used to calculate the evolution of a single atom wavepacket  $|\psi(t)\rangle$ :

$$i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = \hat{H} |\psi(t)\rangle. \quad (1)$$

This equation can be solved in terms of the quantum propagator[18], which can be obtained by the use of a shortcut through the classical limit, and a well-known result of Van Vleck that makes the connection with quantum mechanics.

The Hamiltonian of a single atom with a mass  $m$  in a gravito-inertial field in the classical limit is

$$H = \mathbf{p}^T \boldsymbol{\beta} \mathbf{p} / (2m) - \boldsymbol{\Omega}^T \mathbf{L} - m \mathbf{g}^T \mathbf{q} - m \mathbf{q}^T \boldsymbol{\gamma} \mathbf{q} / 2 + V(t), \quad (2)$$

where  $V(t)$  is the external field,  $\mathbf{q}^T = (x, y, z)$  is the position vector,  $\mathbf{p}^T = (p_x, p_y, p_z)$  is the momentum vector, and  $\mathbf{L}$  is the angular momentum vector of the atom. The gravitational wave and gravitational gradient are represented by the tensors  $\boldsymbol{\beta}$  and  $\boldsymbol{\gamma}$ . The quantity  $\boldsymbol{\Omega}$  is the angular velocity vector of the Earth's rotation, and  $\mathbf{g}$  is the gravitational acceleration vector. The Earth's gravito-inertial field is characterized by the above four parameters:  $\boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\Omega}, \mathbf{g}$ . The superscript  $T$  means the transpose. Some judiciously chosen transformations and choices lead to the following ABCD formalism for the evolution of the variables  $\mathbf{q}$  and  $\mathbf{p}$  from the initial time  $t_a$  to the final time  $t_b$ [18]:

$$\begin{pmatrix} \mathbf{q}_b \\ \mathbf{p}_b/m \end{pmatrix} = \begin{pmatrix} U \boldsymbol{\xi} \\ \mathbf{n}^{-1} U \dot{\boldsymbol{\xi}} \end{pmatrix} + \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \mathbf{q}_a \\ \mathbf{p}_a/m \end{pmatrix}, \quad (3)$$

where  $\mathbf{n}^{-1}$  is the matrix corresponding to the gravitational wave tensor  $\beta$ . Were we to neglect the gravitational wave effect, then  $\mathbf{n}^{-1}$  would be a unit matrix. The matrix  $\mathbf{U}$  corresponds to the rotation term  $-\mathbf{\Omega}^T \mathbf{L}$  in Eq.(2). If there is no angular momentum of the atom,  $\mathbf{U}$  is also a unit matrix. The vector  $\boldsymbol{\xi}$  is the displacement produced by gravity, which describes the classical trajectory of a non-rotating Hamiltonian with the initial conditions  $\boldsymbol{\xi}(t_a) = 0$  and  $\dot{\boldsymbol{\xi}}(t_a) = 0$ . The  $3 \times 3$  matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$  and the vector  $\boldsymbol{\xi}$  can be determined by solving the Hamilton-Jacobi equation[18].

From the above expressions, by neglecting the gravitational wave effect and the rotation, the classical action can be written as

$$\begin{aligned}
S &= S(\mathbf{q}_b, t_b, \mathbf{q}_a, t_a) \\
&= m\dot{\boldsymbol{\xi}} \cdot (\mathbf{q}_b - \boldsymbol{\xi}) + \int_{t_a}^{t_b} L_1(t_1) dt_1 - \int_{t_a}^{t_b} V(t_2) dt_2 \\
&\quad + \frac{m}{2} [(\mathbf{q}_b^T - \boldsymbol{\xi}^T) \mathbf{D} \mathbf{B}^{-1} (\mathbf{q}_b - \boldsymbol{\xi}) - 2(\mathbf{q}_b^T - \boldsymbol{\xi}^T) (\mathbf{B}^{-1})^T \mathbf{q}_a \\
&\quad + \mathbf{q}_a^T \mathbf{B}^{-1} \mathbf{A} \mathbf{q}_a],
\end{aligned} \tag{4}$$

where  $L_1(t)$  is a partial Lagrangian given by  $L_1(t) = m(|\dot{\boldsymbol{\xi}}|^2 + \boldsymbol{\xi}^T \boldsymbol{\gamma} \boldsymbol{\xi} + 2\mathbf{g}^T \boldsymbol{\xi})/2$ . Knowing the classical action allows the determination of the quantum propagator according to Van Vleck's formula[20]:

$$K(\mathbf{q}_b, t_b, \mathbf{q}_a, t_a) = \left(\frac{m}{i\hbar}\right)^{3/2} |\det \mathbf{B}|^{-1/2} \exp(iS_{b,a}/\hbar), \tag{5}$$

where  $\hbar$  is the Plank constant and  $\hbar = h/(2\pi)$ . A complete set of solutions of the Schrödinger equation can be derived from  $K$ . The lowest order Gaussian wavepacket at initial time  $t_a$  is given by

$$\begin{aligned}
\psi(\mathbf{q}, t_a) &= \psi_{wp}(\mathbf{q}, t_a; \mathbf{q}_a, \mathbf{v}_a, \mathbf{X}_a, \mathbf{Y}_a) \\
&= \frac{1}{\sqrt{\det \mathbf{X}_a}} \exp\left[\frac{im}{2\hbar} (\mathbf{q}^T - \mathbf{q}_a^T) \mathbf{Y}_a \mathbf{X}_a^{-1} (\mathbf{q} - \mathbf{q}_a)\right] \\
&\quad \times \exp\left[\frac{im}{\hbar} \mathbf{v}_a \cdot (\mathbf{q} - \mathbf{q}_a)\right],
\end{aligned} \tag{6}$$

which is centered at position  $\mathbf{q}_a$ , has an average velocity  $\mathbf{v}_a = \mathbf{p}_a/m$ , and the complex width parameters  $\mathbf{X}_a, \mathbf{Y}_a$ . Its evolution

$$\begin{aligned}
\psi(\mathbf{q}, t_b) &= \int K(\mathbf{q}, t_b, \mathbf{q}', t_a) \psi_{wp}(\mathbf{q}', t_a; \mathbf{q}_a, \mathbf{v}_a, \mathbf{X}_a, \mathbf{Y}_a) d\mathbf{q}' \\
&= \exp\left[\frac{iS(\mathbf{q}_b, t_b, \mathbf{q}_a, t_a)}{\hbar}\right] \psi_{wp}(\mathbf{q}, t_a; \mathbf{q}_b, \mathbf{v}_b, \mathbf{X}_b, \mathbf{Y}_b)
\end{aligned} \tag{7}$$

is governed by the ABCD matrix formalism for  $\mathbf{q}$  and  $\mathbf{p}$  in Eq.(3) and for  $\mathbf{X}$  and  $\mathbf{Y}$ :

$$\mathbf{X}_b = \mathbf{A}\mathbf{X}_a + \mathbf{B}\mathbf{Y}_a, \quad (8)$$

$$\mathbf{Y}_b = \mathbf{C}\mathbf{X}_a + \mathbf{D}\mathbf{Y}_a. \quad (9)$$

From the ABCD matrix formalism, the problem of the evolution of a single particle wavepacket can be solved by calculating the matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$  and  $\mathbf{D}$ , from which the classical action and the classical equations of motion can be obtained.

As a many-particle system, such as the PCMW, can't be characterized by a single-particle wavepacket, the theory must be extended. A convenient approach to this problem is to carry out a second quantization, which will be done in the following section.

### 3 Tensor ABCD law for the PCMW

In second quantization, atoms can be described by a quantum field operator  $\hat{\psi}(\mathbf{q}, t)$  that satisfies the commutation relations

$$[\hat{\psi}(\mathbf{q}, t), \hat{\psi}^\dagger(\mathbf{q}', t)]_\pm = \delta(\mathbf{q} - \mathbf{q}'), \quad (10)$$

$$[\hat{\psi}(\mathbf{q}, t), \hat{\psi}(\mathbf{q}', t)]_\pm = 0, \quad (11)$$

where  $\hat{\psi}^\dagger$  is the conjugate operator of  $\hat{\psi}$ . The expression  $[\cdots]_-$  is a commutator for bosons, while  $[\cdots]_+$  is an anticommutator for fermions. The field operator  $\hat{\psi}(\mathbf{q}, t)$  is interpreted as annihilating a particle at  $(\mathbf{q}, t)$ , and  $\hat{\psi}^\dagger(\mathbf{q}, t)$  represents the creation of a particle at  $(\mathbf{q}, t)$ . The physical properties of a many-atom gas can be expressed in terms of correlation functions that are expectation values of the field operators, such as the first-order correlation function

$$\Gamma(\mathbf{q}_1, t_1, \mathbf{q}_2, t_2) = \langle \hat{\psi}^\dagger(\mathbf{q}_1, t_1) \hat{\psi}(\mathbf{q}_2, t_2) \rangle, \quad (12)$$

and its higher orders:  $\langle \hat{\psi}^\dagger(\mathbf{q}_1, t_1) \hat{\psi}^\dagger(\mathbf{q}_2, t_2) \cdots \hat{\psi}(\mathbf{q}'_2, t'_2) \hat{\psi}(\mathbf{q}'_1, t'_1) \rangle$ . Here we shall restrict ourselves to the first-order correlation function of bosonic particles in the form of Eq.(12). Its diagonal element  $\Gamma(\mathbf{q}, t, \mathbf{q}, t)$  is the atom number density at position  $\mathbf{q}$  and time  $t$ .

For a completely coherent matter wave, the first-order correlation function can be factorized into the form[12]

$$\langle \hat{\psi}^\dagger(\mathbf{q}_1, t_1) \hat{\psi}(\mathbf{q}_2, t_2) \rangle = \psi^*(\mathbf{q}_1, t_1) \psi(\mathbf{q}_2, t_2), \quad (13)$$

with

$$\psi(\mathbf{q}, t) = \sqrt{N} \langle \mathbf{q} | \phi(t) \rangle, \quad (14)$$

where atoms occupy the same one-particle state  $|\phi(t)\rangle$ , and  $N$  is the total atom number. The degree of coherence

$$g(\mathbf{q}_1, t_1, \mathbf{q}_2, t_2) = \frac{\Gamma(\mathbf{q}_1, t_1, \mathbf{q}_2, t_2)}{\sqrt{\Gamma(\mathbf{q}_1, t_1, \mathbf{q}_1, t_1)} \sqrt{\Gamma(\mathbf{q}_2, t_2, \mathbf{q}_2, t_2)}} \quad (15)$$

is unity for a perfectly coherent matter wave such as the pure BEC, which contains millions of atoms associated with a single-particle wave function.

The ABCD matrix formalism in Sec.2 deals with the evolution of a one-particle wavepacket and is appropriate to describe a coherent matter wave disregarding interactions among atoms [21, 22, 23, 24]. If the coherence of the matter wave is not perfect, that is, if

$$0 < g(\mathbf{q}_1, t_1, \mathbf{q}_2, t_2) < 1, \quad (16)$$

it is not possible to factorize the first-order correlation function into the form of Eq.(13). That means that the matter wave field is only partially coherent and some randomness exists. The field can't just be represented by the single-particle wave function, and second quantized field theory has to be applied.

We concentrate on the propagation and evolution of non-interacting matter waves from a PCMW source. Neglecting the gravitational-wave effects and rotational effects, the single-particle Hamiltonian of Eq.(2) turns out to be

$$H = \mathbf{p}^T \mathbf{p} / (2m) - m \mathbf{g}^T \mathbf{q} - m \mathbf{q}^T \boldsymbol{\gamma} \mathbf{q} / 2. \quad (17)$$

Changing this single-particle Hamiltonian to the second quantized Hamiltonian, the Heisenberg equation of motion for the field operator  $\hat{\psi}(\mathbf{q}, t)$  can be found[25]:

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}(\mathbf{q}, t) = \left( -\frac{\hbar^2 \nabla^2}{2m} - m \mathbf{g}^T \mathbf{q} - \frac{m}{2} \mathbf{q}^T \boldsymbol{\gamma} \mathbf{q} \right) \hat{\psi}(\mathbf{q}, t). \quad (18)$$

It has the same form as the Schrödinger equation for the single-particle wave function, so solution techniques are similar. The propagation of a matter wave field is described by the quantum mechanical propagator

$$\hat{\psi}(\mathbf{q}, t) = \int K(\mathbf{q}, t, \mathbf{q}_0, t_0) \hat{\psi}(\mathbf{q}_0, t_0) d\mathbf{q}_0. \quad (19)$$

The action Eq.(4) can be rewritten in tensor form as

$$\begin{aligned}
S(\mathbf{q}, t, \mathbf{q}_0, t_0) &= m\dot{\boldsymbol{\xi}} \cdot (\mathbf{q} - \boldsymbol{\xi}) + \int_{t_0}^t L_1(t_1) dt_1 - \int_{t_0}^t V(t_2) dt_2 \\
&+ \frac{m}{2} \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{q} - \boldsymbol{\xi} \end{pmatrix}^T \begin{pmatrix} \mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1} \\ \mathbf{C} - \mathbf{D}\mathbf{B}^{-1}\mathbf{A} & \mathbf{D}\mathbf{B}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{q} - \boldsymbol{\xi} \end{pmatrix}
\end{aligned} \tag{20}$$

From Eqs.(5), (12) and (19), we find that the general propagation formula for the first-order correlation function is

$$\Gamma(\mathbf{r}, t) = - \left( \frac{m}{i\hbar} \right)^3 \frac{1}{|\det \mathbf{B}|} \int \Gamma(\mathbf{r}_0, t_0) \exp \left( -i \frac{\Delta S}{\hbar} \right) d\mathbf{r}_0, \tag{21}$$

where  $\mathbf{r}$  and  $\mathbf{r}_0$  are position tensors given by  $\mathbf{r}^T = (\mathbf{q}_1^T, \mathbf{q}_2^T)$ , and  $\mathbf{r}_0^T = (\mathbf{q}_{01}^T, \mathbf{q}_{02}^T)$ ;  $\Delta S = (S_1 - S_2)$  is the action difference, where  $S_j (j = 1, 2)$  is the action from the incident point  $\mathbf{q}_{0j}$  to the output point  $\mathbf{q}_j$ , and can be obtained from Eq.(20) by replacing  $\mathbf{q}$  with  $\mathbf{q}_j$ ,  $\mathbf{q}_0$  with  $\mathbf{q}_{0j}$ . The action difference is

$$\Delta S = m\dot{\boldsymbol{\xi}} \cdot (\mathbf{q}_1 - \mathbf{q}_2) + \frac{m}{2} \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r} - \bar{\boldsymbol{\xi}} \end{pmatrix}^T \mathbf{V} \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r} - \bar{\boldsymbol{\xi}} \end{pmatrix}, \tag{22}$$

where  $\bar{\boldsymbol{\xi}}^T = (\boldsymbol{\xi}^T, \boldsymbol{\xi}^T)$ ,

$$\mathbf{V} = \begin{bmatrix} \bar{\mathbf{B}}^{-1}\bar{\mathbf{A}} & -\bar{\mathbf{B}} \\ \bar{\mathbf{C}} - \bar{\mathbf{D}}\bar{\mathbf{B}}^{-1}\bar{\mathbf{A}} & \bar{\mathbf{D}}\bar{\mathbf{B}}^{-1} \end{bmatrix}, \tag{23}$$

and

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{bmatrix}, \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{0} & -\mathbf{B} \end{bmatrix}, \tag{24}$$

$$\bar{\mathbf{C}} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C} \end{bmatrix}, \quad \bar{\mathbf{D}} = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}. \tag{25}$$

Since  $\Delta S$  is a scalar, we get the relations

$$(\bar{\mathbf{B}}^{-1}\bar{\mathbf{A}})^T = \bar{\mathbf{B}}^{-1}\bar{\mathbf{A}}, \quad (\bar{\mathbf{D}}\bar{\mathbf{B}}^{-1})^T = \bar{\mathbf{D}}\bar{\mathbf{B}}^{-1} \tag{26}$$

$$\bar{\mathbf{C}} - \bar{\mathbf{D}}\bar{\mathbf{B}}^{-1}\bar{\mathbf{A}} = -(\bar{\mathbf{B}})^T, \tag{27}$$

among the sub-matrices of Eq.(23). These will be used in derivations that follow.

In conventional optics, the cross-spectral density of a partially coherent wave is usually described by the Gaussian-Schell-model(GSM)[15] expression

$$W(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \sqrt{S(\boldsymbol{\rho}_1)}\sqrt{S(\boldsymbol{\rho}_2)}g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2), \quad (28)$$

where  $S(\boldsymbol{\rho})$  represents the spectral density, and  $g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$  represents the spectral degree of coherence, given by

$$S(\boldsymbol{\rho}) = G_0 \exp[-\frac{1}{2}\boldsymbol{\rho}^T(\boldsymbol{\sigma}_s^2)^{-1}\boldsymbol{\rho}], \quad (29)$$

$$g(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \exp[-\frac{1}{2}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)^T(\boldsymbol{\sigma}_g^2)^{-1}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)], \quad (30)$$

where

$$\boldsymbol{\rho}_1^T = (x_1, y_1), \quad \boldsymbol{\rho}_2^T = (x_2, y_2) \quad (31)$$

are position vectors of two arbitrary points in the transverse plane.  $G_0$  is a positive quantity.  $\boldsymbol{\sigma}_s$  represents the transverse spot width and  $\boldsymbol{\sigma}_g$  represents the transverse coherent width.  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_g$  are  $2 \times 2$  matrices with transpose symmetry.

This partial coherence theory in optics can be extended to describe the PCMW according to the analogy between the Schrödinger equation (18) of a matter wave and the paraxial wave propagation equation of light

$$\nabla_T^2 A(\mathbf{q}) + 2ik \frac{\partial A(\mathbf{q})}{\partial z} = 0, \quad (32)$$

where  $A(\mathbf{q})$  is the amplitude of the electric field as expressed by  $E(\mathbf{q}, t) = A(\mathbf{q}) \exp[i(kz - \omega t)]$ , and  $k = 2\pi/\lambda$  is the modulus of the wave vector. The squared transverse Laplacian  $\nabla_T^2$  is mathematically analogous to the atomic kinetic energy  $-(\hbar^2/2m)\nabla^2$ . The difference between Eq.(18) and Eq.(32) is that Eq.(18) has a time derivative while the paraxial wave equation Eq.(32) has a spatial derivative along the propagation axis  $z$ . The GSM partially coherent light as described by Eq.(28) has two spatial dimensions perpendicular to the propagation axis  $z$ , while the GSM of the PCMW may have three spatial dimensions. The first-order correlation function of a GSM matter wave can be expressed as

$$\Gamma(\mathbf{q}_1, \mathbf{q}_2) = G_0 \exp \left\{ -\frac{1}{4} [\mathbf{q}_1^T(\boldsymbol{\sigma}_s^2)^{-1}\mathbf{q}_1 + \mathbf{q}_2^T(\boldsymbol{\sigma}_s^2)^{-1}\mathbf{q}_2] - \frac{1}{2}(\mathbf{q}_1 - \mathbf{q}_2)^T(\boldsymbol{\sigma}_g^2)^{-1}(\mathbf{q}_1 - \mathbf{q}_2) \right\}, \quad (33)$$

where both the atom number density and the coherence degree have Gaussian distributions. The parameters  $G_0$ ,  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_g$  are related to the temperature of the atom gas.



Equation (33) can be rewritten as

$$\Gamma(\mathbf{r}) = G_0 \exp \left( \frac{im}{2\hbar} \mathbf{r}^T \mathbf{M}_i^{-1} \mathbf{r} \right), \quad (34)$$

with  $\mathbf{r}^T = (\mathbf{q}_1, \mathbf{q}_2)$ . The tensor

$$\mathbf{M}_i^{-1} = \begin{bmatrix} \frac{i\hbar}{2m}(\boldsymbol{\sigma}_s^2)^{-1} + \frac{i\hbar}{m}(\boldsymbol{\sigma}_g^2)^{-1} & -\frac{i\hbar}{m}(\boldsymbol{\sigma}_g^2)^{-1} \\ -\frac{i\hbar}{m}(\boldsymbol{\sigma}_g^2)^{-1} & \frac{i\hbar}{2m}(\boldsymbol{\sigma}_s^2)^{-1} + \frac{i\hbar}{m}(\boldsymbol{\sigma}_g^2)^{-1} \end{bmatrix} \quad (35)$$

is a  $6 \times 6$  symmetric matrix, which may be called the complex curvature tensor of the PCMW; where  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_g$  are  $3 \times 3$  matrices with transpose symmetry, given by

$$\boldsymbol{\sigma}_s = \begin{bmatrix} \sigma_{sxx} & \sigma_{sxy} & \sigma_{sxz} \\ \sigma_{syx} & \sigma_{syy} & \sigma_{syx} \\ \sigma_{sxx} & \sigma_{sxy} & \sigma_{sxx} \end{bmatrix}, \quad (36)$$

$$\boldsymbol{\sigma}_g = \begin{bmatrix} \sigma_{gxx} & \sigma_{gxy} & \sigma_{gxz} \\ \sigma_{gyx} & \sigma_{gyy} & \sigma_{gyz} \\ \sigma_{gxx} & \sigma_{gxy} & \sigma_{gxx} \end{bmatrix}. \quad (37)$$

The tensor expression can be used to describe a general matter wave system including the asymmetric and anisotropic systems, and is very convenient for its compactness. The matrix  $\boldsymbol{\sigma}_s$ , representing widths of the matter wave in 3D space, characterizes the spatial size of the matter wave. The matrix  $\boldsymbol{\sigma}_g$  describes the coherent length of the matter wave. When  $\boldsymbol{\sigma}_g$  decreases to zero, it corresponds to a completely incoherent matter wave. When  $\boldsymbol{\sigma}_g$  increases to infinity, the matter wave can be treated as a completely coherent matter wave.

Substituting Eq.(22) and Eq.(34) into Eq.(21), we get

$$\begin{aligned} \Gamma(\mathbf{r}, t) &= -\left(\frac{m}{i\hbar}\right)^3 \frac{G_0}{|\det \mathbf{B}|} \\ &\cdot \int \exp \left[ \frac{im}{2\hbar} \left( Q + 2\dot{\boldsymbol{\xi}} \cdot (\mathbf{q}_1 - \mathbf{q}_2) \right) \right] d\mathbf{r}_0, \end{aligned} \quad (38)$$

where

$$\begin{aligned} Q &= (\mathbf{r} - \bar{\boldsymbol{\xi}})^T (\bar{\mathbf{C}} + \bar{\mathbf{D}} \mathbf{M}_i^{-1}) (\bar{\mathbf{A}} + \bar{\mathbf{B}} \mathbf{M}_i^{-1})^{-1} (\mathbf{r} - \bar{\boldsymbol{\xi}}) \\ &+ \left| (\bar{\mathbf{B}}^{-1} \bar{\mathbf{A}} + \mathbf{M}_i^{-1})^{1/2} \mathbf{r}_0 - (\bar{\mathbf{B}}^{-1} \bar{\mathbf{A}} + \mathbf{M}_i^{-1})^{-1/2} \bar{\mathbf{B}}^{-1} (\mathbf{r} - \bar{\boldsymbol{\xi}}) \right|^2. \end{aligned} \quad (39)$$

In the derivation of Eq.(39), the transpose-symmetric property of  $\mathbf{M}_i^{-1}$  and Eq.(26) have been used. Integrating Eq.(38), we have

$$\begin{aligned}\Gamma(\mathbf{r}) &= G_0[\det(\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{M}_i^{-1})]^{-1/2} \exp \left[ im\dot{\boldsymbol{\xi}} \cdot (\mathbf{q}_2 - \mathbf{q}_1)/\hbar \right] \\ &\times \exp \left[ \frac{im}{2\hbar} (\mathbf{r} - \bar{\boldsymbol{\xi}})^T \mathbf{M}_f^{-1} (\mathbf{r} - \bar{\boldsymbol{\xi}}) \right],\end{aligned}\quad (40)$$

where  $\mathbf{M}_i^{-1}$  and  $\mathbf{M}_f^{-1}$  denote the complex curvature tensors of the PCMW at the initial and the final time, respectively. They satisfy the condition

$$\mathbf{M}_f^{-1} = (\bar{\mathbf{C}} + \bar{\mathbf{D}}\mathbf{M}_i^{-1})(\bar{\mathbf{A}} + \bar{\mathbf{B}}\mathbf{M}_i^{-1})^{-1}. \quad (41)$$

In the derivation of Eq.(40), the integral formula  $\int_{-\infty}^{\infty} \exp[-ax^2]dx = \sqrt{\pi/a}$  has been used. Equation (41) may be called the tensor ABCD law for the PCMW.

## 4 Evolution of the PCMW in a gravitational field

To illustrate the usage of the formulas derived above, we are going to calculate the atom density distribution of the PCMW in a transverse plane in a gravitational field, which is the diagonal element of the first-order correlation function  $\Gamma(\mathbf{q}, \mathbf{q})$ . The object we consider is a matter wave of cold  $Rb^{87}$  atoms. It propagates along the  $z$  direction, and the gravitational acceleration  $\mathbf{g}$  with a linear gradient  $\gamma$  goes in the same direction. As is the case in Ref.[22], the cold atom gas is prepared in a magnetic trap with trapping frequencies of  $\omega_x = \omega_z = 2\pi \times 330$  Hz and  $\omega_y = 2\pi \times 8$  Hz. The initial temperature of the  $Rb^{87}$  atom gas is assumed to be the recoil temperature by laser cooling [8]:  $T_{rc} = 362$  nk.

The propagation matrices  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{D}$  are

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & A_z \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} t - t_0 & 0 & 0 \\ 0 & t - t_0 & 0 \\ 0 & 0 & B_z \end{pmatrix}, \quad (42)$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_z \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D_z \end{pmatrix}, \quad (43)$$

where  $A_z = D_z = \cosh[\sqrt{\gamma}(t - t_0)]$ ,  $B_z = \sinh[\sqrt{\gamma}(t - t_0)]/\sqrt{\gamma}$ , and  $C_z = \gamma B_z$ . The gravitational displacement vector is  $\boldsymbol{\xi}^T = (0, 0, \xi_z)$ , and  $\xi_z = (g/\gamma)[1 - \cosh(\sqrt{\gamma}(t - t_0))]$  [18].

Assuming the ultracold atoms are initially in a magnetic quadratic trap, the widths of the atom gas in the  $x, y, z$  directions are[2]

$$\sigma_{sj} = \left( \frac{k_B T}{m \omega_j^2} \right)^{1/2}, \quad (44)$$

where  $\omega_j$  ( $j = x, y, z$ ) is the harmonic oscillator frequency in each dimension. Substituting the initial temperature  $T_{rc}$  and the initial trapping frequencies  $\omega_j$  into the expression of  $\sigma_{sj}$ , the widths of the matter wave source can be obtained:  $\sigma_{sxx} = 2.84 \mu\text{m}$ ,  $\sigma_{syy} = 117.25 \mu\text{m}$ . As we have assumed that the ultracold atom gas propagates continuously along the  $z$  direction, the width along the  $z$  axis is infinite:  $\sigma_{szz} = \infty$ . Then we can write out the width tensor of the matter wave source as

$$(\sigma_s^2)^{-1} = \begin{pmatrix} \sigma_{sx}^{-2} & 0 & 0 \\ 0 & \sigma_{sy}^{-2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{mm}^{-2}. \quad (45)$$

We assume the coherent length  $l_c$  to be  $100\lambda_T$ , which is between the thermal de Borgile wavelength  $\lambda_T = \sqrt{2\pi\hbar^2/(mK_B T)}$  and the infinity. The coherent length matrix is

$$(\sigma_g^2)^{-1} = \begin{pmatrix} l_c^{-2} & 0 & 0 \\ 0 & l_c^{-2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{mm}^{-2}. \quad (46)$$

Substituting the width matrix  $(\sigma_s^2)^{-1}$  and the coherent length matrix  $(\sigma_g^2)^{-1}$  into Eq.(35), we can calculate the final complex curvature tensor  $\mathbf{M}_f^{-1}$  from Eq.(41) and get the atom density distributions from Eq.(40). The expression of **A**, **B**, **C** and **D** in Eq.(42) shows that the expansion of the matter wave is dependent on time, and the transverse density distributions at different temporal points reveal the evolution of the matter wave as it propagates.

Figure 1 shows the evolution of the PCMW, with the parameters given in the caption. The initial atom density profile, which is shown in Fig.1(a), is elliptical due to the initial inhomogeneous confinement of the cold atom gas on the  $x$ - $y$  plane; that is, the confinement in the  $x$ -direction is stronger, which causes a larger speed of expansion of the matter wave in this direction when the confinement is removed. During evolution without confinement, the atom density profile expands gradually ( Fig1(b)), and changes to a circle (Fig.1(c)). Finally the atom density profile becomes an ellipse again with the major axis perpendicular to that of the initial ellipse, as shown in Fig.1(d). This is because the degree of expansion in the  $x$ -direction exceeds that in the  $y$ -direction. This property is similar to that of light beams.

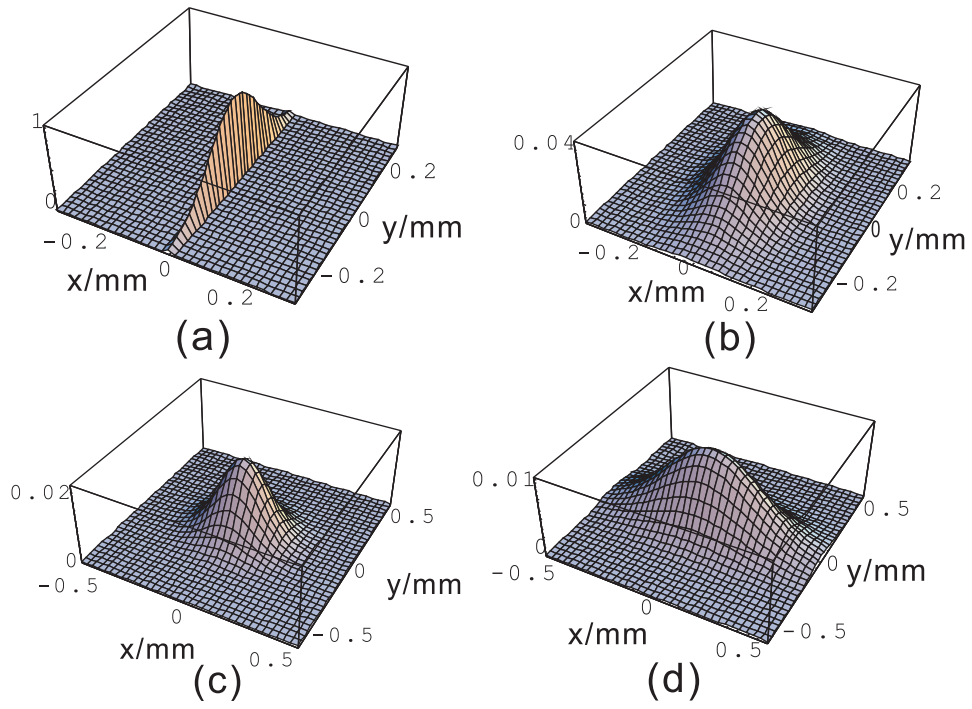


Figure 1: The transverse relative atom density distribution for an anisotropic GSM PCMW at different propagating times  $t$ . (a)  $t = 0$  s; (b)  $t = 0.5$  s; (c)  $t = 1$  s; (d)  $t = 2$  s.

## 5 Conclusion

In this paper, a theoretical description of a partially coherent matter wave (PCMW) based on the analogy between a matter wave and an optical wave is presented. The propagation and evolution formula of the Gaussian-Schell-Model PCMW is provided, and a generalized tensor ABCD law is derived. As an example, we analyzed the atom density profile in a transverse plane of a  $Rb^{87}$  cold atom beam. The results show that the tensor ABCD law is a very convenient method for treating the propagation and transformation of partially coherent cold atom beams. Most importantly, the previous results for partially coherent light can be converted into a PCMW. Our method can be applied to analyze the propagation and evolution properties of ultracold atom beams in various potentials, such as an optical potential and a magnetic potential. This method can also be extended to pulsed matter wave propagation, and to include the interactions among atoms and the internal state of atoms with some modifications. These will be the topics of further investigations.

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